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An Efficient Numerical Scheme for a Baroclinic Quasi-Geostrophic Model

SAMUEL Y. K. YEE

Air Force Cambridge Research Laboratories, Bedford, Massachusetts 01730

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An efficient numerical scheme for solving a multilevel geostrophic forecast model having consistent first-order approximations is presented. The approach involves eliminating the vertical velocity between the vorticity and thermodynamic equations to yield a three-dimensional prognostic equation, reducing this three-dimensional equation to two-dimensional by an orthogonal transformation, and solving the transformed system by an iterative method. The scheme is shown to be at least four times as efficient as another popular scheme. Time integrations using NMC-analysed grid-point geopotential as initial data are carried out for calibration purposes. Finally, advantages and implications of such a scheme are discussed.

I. INTRODUCTION

A popular numerical approach in solving the geostrophic forecast system is to obtain first a so-called omega-equation by eliminating the time-dependent terms between the vorticity and the thermodynamic equations, and then solve this diagnostic equation together with the vorticity equation numerically. This method (Method A) involves therefore, the inversion of a three-dimensional and a two-dimensional elliptic system at each time-step during the course of a time integration. An alternate approach (Method C) is to eliminate the vertical velocity between the vorticity and thermodynamic equations, yielding a three-dimensional elliptic-hyperbolic system. The problem of time integration then becomes that of seeking solutions to this three-dimensional system at each time step.

From the computational point of view, Method C is about twice as efficient as Method A because it involves the inversion of only one elliptic system. Charney, Gilchrist, and Shuman (1956) discussed this method in some detail and presented numerical results for several test cases each having slightly different assumptions. Phillips (1956), in his now classical numerical experiments, employed essentially the same approach in a two-level quasi-geostrophic model. The purpose of this paper is to show that for an energetically consistent quasi-geostrophic model, where the static stability of the atmosphere is a

function of pressure only, the efficiency of Method C can be increased by another factor of two. This scheme (Method B) involves writing a system of linear algebraic equations in vector form, applying an orthogonal transformation to this system, and solving the transformed system by relaxation. As it stands, this scheme is applicable to geostrophic models such as the one in use at the Air Force Global Weather Central (Palucci, 1970).

II. The Model

The model considered here is a quasi-geostrophic model having consistent first-order approximations. With the following assumptions

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \tag{1}$$

$$T = -\frac{pf_0}{R}\frac{\partial \psi}{\partial p}, \qquad \sigma = \frac{\kappa \overline{T}}{p} - \frac{\partial \overline{T}}{\partial p}, \tag{2}$$

the large-scale motions in a frictionless, adiabatic, dry atmosphere in hydrostatic equilibrium may be described by

$$\nabla^2 \frac{\partial \psi}{\partial t} = -J(\psi, \nabla^2 \psi + f) + f_0 \frac{\partial \omega}{\partial p}, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial p}\right) = -J\left(\psi, \frac{\partial \psi}{\partial p}\right) - \frac{R\sigma}{f_0 p}\omega. \tag{4}$$

This is an initial-boundary value problem and may be solved in a closed domain with proper boundary conditions. As mentioned earlier, two methods, designated as Methods A and C here, have been used in solving this problem. We shall outline here only the approach of Method B.

Discretization of (3) and (4) in the vertical gives

$$\nabla^2 \left(\frac{\partial \psi}{\partial t}\right)_k = -J(\psi_k, \nabla^2 \psi_k + f) + f_0(\omega_k - \omega_{k-1})/\Delta p, \tag{5}$$

$$\left(\frac{\partial \psi}{\partial t}\right)_{k+1} - \left(\frac{\partial \psi}{\partial t}\right)_k = -J(\psi_k{}^*, \psi_{k+1} - \psi_k) - \frac{R\sigma_k}{f_0 p_k{}^*}\Delta p \omega_k, \tag{6}$$

where $\psi_k{}^*$ is the stream function at pressure $p_k{}^* = (p_k + p_{k+1})/2$, and will be approximated by $(\psi_k + \psi_{k+1})/2$. Definition of the model variables in the vertical

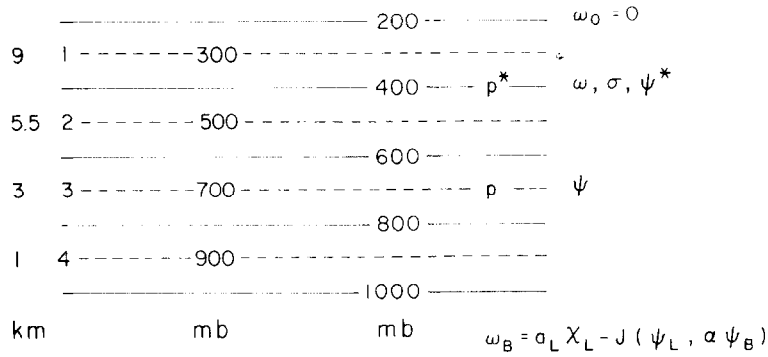

FIG. 1. Vertical domain and resolution of the model.

is shown in Fig. 1. Note that ψ_k is defined only at p_k, and ψ_k^*, σ_k, ω_k only at p_k^*. Defining

$$\chi_k = \left(\frac{\partial \psi}{\partial t}\right)_k,$$
$$W_k = f_0 \omega_k / \Delta p,$$
$$1/a_k = (\Delta p)^2 R\sigma_k/(f_0^2 p_k^*),$$

and imposing the boundary conditions

$$\begin{aligned} W_0 &= 0, \\ W_K &= a_K \chi_K - J(\psi_K, a_K \psi_B) \end{aligned} \tag{7}$$

(5) and (6) can be written for a K-layer model as

$$\nabla^2 \chi_k = -J(\psi_k, \nabla^2 \psi_k + f) + W_k - W_{k-1} \tag{8}$$

$$\chi_{k+1} - \chi_k = -J(\psi_k, \psi_{k+1}) - W_k/a_k \tag{9}$$

Here ψ_B is the stream function at the lower boundary. Substitution of W_k from (9) into (8) will give

$$\begin{aligned} \nabla^2 \chi_k + \{a_{k-1}\chi_{k-1} - (a_{k-1} + a_k)\chi_k + a_k \chi_{k+1}\} \\ = -J(\psi_k, \nabla^2 \psi_k + f + a_{k-1}\psi_{k-1} + a_k \psi_{k+1}) \end{aligned} \tag{10}$$

where $a_0 = 0$, $\chi_{K+1} = 0$, and ψ_{K+1} will have to be specified. We shall assume here $\psi_{K+1} = \psi_B$. To discretize in the horizontal, we define $\alpha_k = 4d^2 a_k$ and

$$\nabla^2 = \frac{4d^2}{m^2}\nabla^2, \qquad \mathbf{J} = \frac{4d^2}{m^2} J, \tag{11}$$

then Eq. (10) can be written in vector notation for each grid point (i, j) in a computational domain Γ as

$$\nabla_{ij}^2 \boldsymbol{\chi} - \frac{1}{m_{ij}^2} A \boldsymbol{\chi}_{ij} = \mathbf{F}_{ij}, \tag{12}$$

where

$$\boldsymbol{\chi}_{ij} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_K \end{pmatrix}_{ij} \qquad A = \begin{pmatrix} \alpha_1 & -\alpha_1 & & \\ -\alpha_1 & (\alpha_1 + \alpha_2) & -\alpha_2 & \\ & & \cdots & \\ & & -\alpha_{K-1} & (\alpha_{K-1} + \alpha_K) \end{pmatrix}$$

$$-\mathbf{F}_{ij} = \begin{bmatrix} \mathbf{J}_{ij}\left(\psi_1, \frac{m^2}{d^2}\nabla^2\psi_1 + f + a_1\psi_2\right) \\ \mathbf{J}_{ij}\left(\psi_2, \frac{m^2}{d^2}\nabla^2\psi_2 + f + a_1\psi_1 + a_2\psi_3\right) \\ \cdots \\ \mathbf{J}_{ij}\left(\psi_K, \frac{m^2}{d^2}\nabla^2\psi_K + f + a_{K-1}\psi_{K-1} + a_K\psi_B\right) \end{bmatrix}.$$

Note that $\mathbf{F}_{ij}$ is a known function of the spatial coordinates, and the system is complete if lateral boundary conditions for $\boldsymbol{\chi}$ are specified. Since the matrix A is real symmetric, we may solve (12) by applying an orthogonal transformation followed by a two-dimensional relaxation. Details of the method of solution are given in the next section.

III. Method of Solution

To solve (12), we note that the matrix A is tridiagonal. Therefore, the only coupling of any component of the unknown vector $\boldsymbol{\chi}_{ij}$ in (12) is with its neighboring components through the two nonzero off-diagonal elements of A. Furthermore, A is real symmetric and is orthogonally similar to a real diagonal matrix D. Thus there exists a nonsingular matrix P such that

$$PAP^{-1} = D. \tag{13}$$

Here $D = (\lambda_k \delta_{kl})$, δ_{kl} being the Kronecker delta, is a diagonal matrix composed of the K distinct eigenvalues λ_k of A, and P is a $K \times K$ orthogonal matrix composed of the normalized eigenvectors of A.

We now define $\boldsymbol{\xi}_{ij} = (\xi_1, \xi_2, \ldots, \xi_K)_{ij}^T$, and let

$$\boldsymbol{\chi}_{ij} = P^{-1}\boldsymbol{\xi}_{ij}, \tag{14}$$

(12) can then be written as

$$\nabla_{ij}^2 \boldsymbol{\xi} - \frac{1}{m_{ij}^2} D\boldsymbol{\xi}_{ij} = P\mathbf{F}_{ij} \equiv \mathbf{G}_{ij}\,, \tag{15}$$

with $\boldsymbol{\xi}_{ij}$ specified on the lateral boundary. Since D is diagonal, the components of $\boldsymbol{\xi}_{ij}$ in (15) are uncoupled. In fact, (15) can be written in scalar form for all (i, j) in Γ as

$$\nabla_{ij}^2 \xi_k - \frac{\lambda_k}{m_{ij}^2} \xi_{ijk} = g_{ijk}\,, \tag{16}$$

with ξ_{ijk} specified on the lateral boundary. This is a two-dimensional Helmholtz equation and may be solved using a large variety of iterative methods. Once $\boldsymbol{\xi}_{ij}$ is known for all (i, j) in Γ, $\boldsymbol{\chi}_{ij}$ can be obtained simply through (14). The fact that A is real symmetric here makes its eigenvalue problem a trivial one. Furthermore, the transformation matrix P is orthogonal and $P^{-1} = P^T$, and we have not even the problem of inverting the matrix P. In fact, since $\sigma(p)$ is time independent, so is the matrix A. Thus its eigenvalues λ_k and their eigenfunctions need only be computed once for all time.

After the values of $\boldsymbol{\chi}_{ij}$ have been computed for all (i, j) in Γ, time integration can be performed using any stable time-differencing scheme. A two-step Lax–Wendroff scheme has been adopted in the present study. The routine consists of computing first values of ψ_{ij} at half time-step using forward time-differencing, then tendencies at half time-step using these values of ψ_{ij}, and finally values at full time-step using centered time-differencing. Specifically (e.g., Richtmyer and Morton, 1967),

$$\psi_{ij}^{n+1/2} = (\psi_{i-1j}^n + \psi_{i+1j}^n + \psi_{ij-1}^n + \psi_{ij+1}^n)/4 + 0.5\,\Delta t\,\chi_{ij}^n\,,$$

$$\psi_{ij}^{n+1} = \psi_{ij}^n + \Delta t\,\chi_{ij}^{n+1/2}.$$

Here the superscript n indicates time steps with increment Δt.

IV. Efficiency Comparison

To compare the efficiency of this method with that of Method A, we shall use as a measure the number of arithmetic operations required in each method. The complexity of the problem makes a precise operational count impractical and is therefore not attempted. Instead, we shall make liberal use of assumptions and present only relative comparisons.

Since the NMC grid is octagonal and is not readily subjected to analysis, we shall consider here an "equivalent" square grid with roughly the same number of interior grid-points as the NMC grid. Furthermore, we shall consider here only the point successive overrelaxation (PSOR) method. Other iterative methods may change the overall efficiency of both schemes but should not affect the relative comparison to any significant degree.

For the kind of boundary-value problems considered here, the asymptotic convergence rate of the PSOR method is (e.g., Varga, 1962).

$$R_\infty = -\ln \mu,$$

where μ is the spectral radius of the iterative matrix. Assuming that the average rate of convergence is as fast as the asymptotic rate, the number of iterations necessary for reduction of any error vector by ϵ orders of magnitude is given by

$$N = -\frac{\ln 10^{-\epsilon}}{R_\infty} = \frac{2.3\epsilon}{R_\infty}.$$

In the case of a two-dimensional Poisson equation over a square region having $(I \times I)$ equally spaced interior grid-points, for large I, the asymptotic convergence rate is well known to be

$$R_\infty \sim \frac{2\pi}{I+1}.$$

And therefore,

$$N_2 \sim 0.366\epsilon(I+1).$$

For a three-dimensional Poisson equation in a domain having $(I \times I \times K)$ interior grid-points, the spectral radius of the Jacobi matrix of the system is (e.g., Young, 1954)

$$\mu(J) = 1 - \frac{3}{2(I+1)^2 + (K+1)^2}\frac{\pi^2}{2}.$$

And, therefore,

$$R_\infty \sim \left[\frac{3}{2(I+1)^2 + (K+1)^2}\right]^{1/2} 2\pi, \tag{17}$$

$$N_3 \sim 0.366\epsilon \left[\frac{2(I+1)^2 + (K+1)^2}{3}\right]^{1/2}. \tag{18}$$

The omega equation is not strictly a three-dimensional Poisson equation because of the presence of the map factor m and the static stability factor σ. However,

consideration of the spatial dependence of m and σ would render the rate of convergence extremely difficult to appraise. We shall assume here that (17) and (18) are applicable to the omega equation. Finally, a Helmholtz equation of the form of (16) applied to an $(I \times I)$ grid, again for large I and ignoring the variation of the map factor, has

$$R_\infty \sim [1 + \lambda/2]^{1/2} \frac{2\pi}{I+1},$$

$$N_\lambda \sim 0.366(I+1)\left[\frac{2}{2+\lambda}\right]^{1/2} \epsilon.$$

With rough estimates of the convergence rates available, specific comparisons may now be made. Before proceeding further, however, it should be remembered that for a computational domain having $(I \times I \times K)$ interior grid points, each iteration involves computations at K times $(I \times I)$ grid points. In effect, the number of grid points involved in one iteration is equal to that in K two-dimensional iterations. Moreover, at each grid point, the ratio of the number of arithmetic operations of a three-dimensional iteration to a two-dimensional one is about 1.25. Thus in comparing the computing effort of the two, we must weight the number of three-dimensional iterations by a factor of $1.25K$.

For a K-layer model, Method A requires at each time-step the solution of one three-dimensional and K two-dimensional Poisson systems. The total number of two-dimensional iterations per time step is then

$$N_A \sim 0.366\epsilon K\left(1.25\left[\frac{2(I+1)^2 + (K+1)^2}{3}\right]^{1/2} + (I+1)\right).$$

For $I = 42$, $K = 4$,

$$N_A \sim 1.46\epsilon(44 + 43) \sim 128\epsilon.$$

With Method B,

$$N_B \sim 0.366\epsilon(I+1)\sum_{k=1}^{4}\left(\frac{2}{2+\lambda_k}\right)^{1/2}.$$

For a model having vertical resolution shown in Fig. 1, typical values of λ_k are those given in Table 1. We have then

$$N_B \sim 0.366\epsilon(I+1) \times 1.8 \sim 28\epsilon.$$

The relative amount of work demanded by the two methods is given by the ratio

$$\frac{N_A}{N_B} \sim \frac{128\epsilon}{28\epsilon} \sim 4, \qquad \epsilon \neq 0.$$

TABLE 1

Values of σ and α used in the test computations[a]

p^*(mbar)	$\sigma(K\ \text{mbar}^{-1})$	α	λ
400	0.10	1.977	1.286
600	0.06	5.324	6.180
800	0.04	10.403	18.511
1000	0.04	26.007	43.698

[a] Values of λ are for 300-, 500-, 700-, and 900 mbar.

We see here that Method B is about four times as efficient as Method A. In the test computations described in Section V, it takes about 10 min for a CDC 6600 to produce a 24-hr forecast using Method A, and about 3 min using Method B. The ratio is thus roughly three to one. With the exception of using the optimal relaxation coefficients, no other optimization has been attempted in these computations.

V. Test Computations

The method presented here has been programmed to cary out test computations using analysed data for the NMC-grid reproduced in Fig. 2. For programming convenience, test computations have been done only with a four-level model of the troposhere. The vertical resolution of the model is shown in Fig. 1. Here the atmosphere below 200 mbar is divided into four equal layers by mass. The stream function is defined at pressures indicated by the dashed lines at 300-, 500-, 700-, and 900 mbar. σ and ω are defined at pressures represented by full lines at 400,-600-, 800-, and 1000 mbar. For simplicity, the lateral boundary has been assumed to be a rigid wall. Specifically, $\partial\psi/\partial x$, $\partial\psi/\partial y$, $\partial\psi/\partial t$ are required to vanish at all times on the lateral boundary. Values of σ adopted in the computations are given in Table 1. System (12) is complete if ψ_B is specified. We have assumed ψ_B to be time invariant here and having the initial values of ψ at 1000 mbar.

Figure 3 shows a sample 500 mbar 24-hr prediction, using the NMC-analyzed 00Z 1 November 1969 grid point data as initial conditions and a time increment of 4 hr. Here the contours are plotted in decameters. Qualitatively, the forecast movements of the weather systems in this case are too slow when compared with observations at verification time. This in no way reflects the accuracy of this scheme, however, because other features in the model could well account for these errors. At any rate, our interest in the accuracy should be limited here to relative comparisons between Methods A and B.

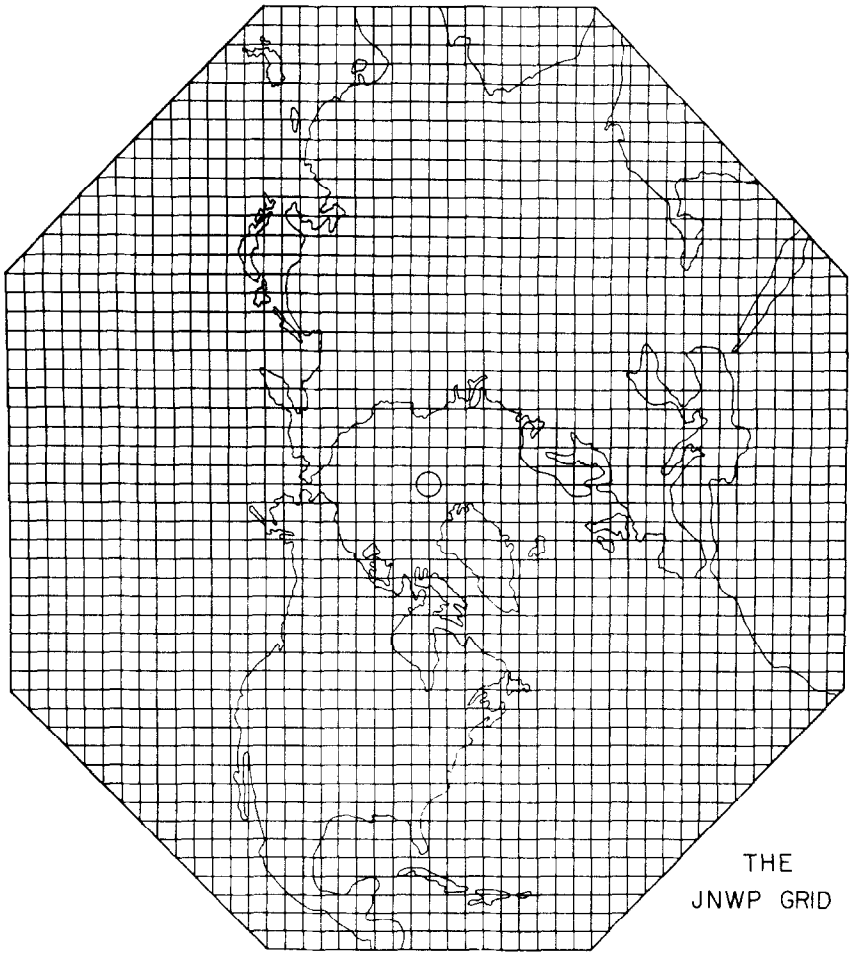

FIG. 2. Horizontal domain and resolution of the model.

Error analyses of a complicated system such as this are difficult and have not been attempted. Empirical tests have been conducted in order to compare the relative accuracy of the two methods. This is accomplished by solving the system (3) and (4) using both methods and then computing the statistics of the predicted stream function fields. Judging from the values of R_{AB} in Table 2, it is evident that the two schemes give highly correlated results. Here the correlation coefficients R_{AB} and normalized mean-square differences S_{AB} of the 24-hr predicted stream function fields for 00Z 1 November 1969 are given for each of the forecast levels. They are measures of the relative reproducibility of the two schemes. The values

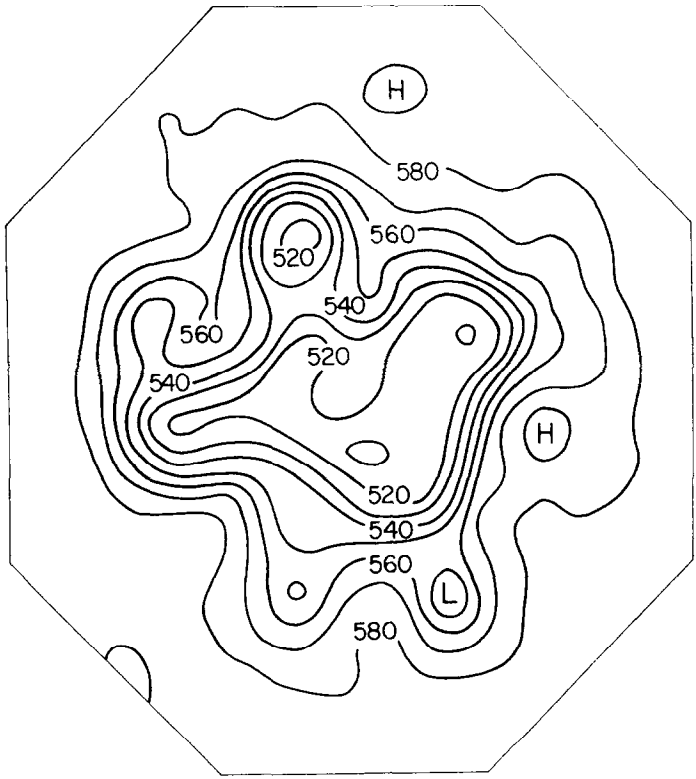

FIG. 3. A sample 500 mb 24-hr predicted height field.

of the normalized mean-square differences require further clarification. They are normalized by the variances of stream function fields at the level in question. Thus $R_{AB} = 1$, $S_{AB} = 0$ suggests, for example, that both methods A and B produce identical results. And $R_{AB} = 0$, $S_{AB} = 2$ indicates that the results given by the two methods are completely uncorrelated, with mean-square differences as large as the variances of the predicted fields. When the mean-square differences are larger than the variances of the fields, S_{AB} will, of course, take on values larger than 2. The deterioration of R_{AB} and S_{AB} at the lowest forecast level in Table 2 could be attributed to the fact that in computations using Method A, the second term in the lower boundary condition in (7) had to be dropped. However, contributions due to other factors cannot be ruled out at the moment. As a matter of incidental interest, the 24-hr predicted fields were also correlated with the observed fields at verification time, 00Z 2 November 1969. Values of R and S in Table 2 seem to indicate that, as formulated, the forecast skill of Method B is at least equal to that of Method A.

TABLE 2

Correlation coefficients R, and normalized mean-square differences S between two stream function fields[a]

p	R_{AB}	S_{AB}	R_{AO}	S_{AO}	R_{BO}	S_{BO}
300	1.00	0.00	0.95	0.10	0.97	0.06
500	0.99	0.01	0.94	0.12	0.97	0.07
700	0.97	0.06	0.90	0.18	0.96	0.08
900	0.89	0.22	0.77	0.42	0.92	0.15

[a] AB: between fields computed via methods A and B, similarly for AO and BO, where O denotes observed fields.

VI. Discussion

We have shown that for an energetically consistent quasi-geostrophic model, both Methods A and B give very similar results and yet the latter is about four times as efficient. If the variation of the map factor can be ignored, we may solve the two-dimensional Helmholtz equation (16) over a rectangle directly by performing another orthogonal transformation. This would further increase the efficiency of Method B by another factor of four (Yee, 1969). While one might not wish to do this when dealing with models used for operational purposes, this approach does provide us with an efficient way to conduct numerical experiments.

Even in the cases where there are reasons not to solve the system by direct inversion, a two-dimensional problem is numerically easier to handle than its three-dimensional counterpart. This shows up, for example, both in estimating the optimum relaxation coefficients and in using other iterative methods for solution. Another advantage of the latter method lies in the simplicity of the form of the forcing function. The forcing function $\mathbf{F}_{ij}$ in (12) involves only the Jacobian of known spatial functions. This is important on two counts. First, a difference analog of the Jacobian having desirable characteristics is available for long-term integration (Arakawa, 1966), and second, it is readily amendable to forms which permit the use of higher order time-differencing schemes.

Finally, the divergence term $f_0 \, \partial\omega/\partial p$ in the vorticity equation is separated in Method B explicitly into two parts, that due to thickness advection and that due to local change of ψ in a given layer as well as in its neighboring layers. The latter, appearing as the bracketed terms in (10) may be considered as the counterpart of an "artificial divergence term" introduced by Cressman (1963) in the vorticity equation in his model. Although this was done out of operational needs to suppress the retrogression of atmospheric long waves, we can now see that it is in fact

justifiable. This is even more obvious in (16) because, though written in a different coordinate system, this equation is now identical in form to Cressman's vorticity equation. Thus, the isolation of an explicit time-dependent component in the divergence term provides added insight to Cressman's correction term to the local stream function tendency in his three-level model.

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